Solutions to Question Sheet 3, Limits III. v1. 2019-20

Divergence

1. i) Let f be defined on a right-hand open interval of $a \in \mathbb{R}$ (i.e. on $(a, a+\eta)$ for some $\eta > 0$). Write out the K - δ definition for

$$\lim_{x \to a+} f(x) = +\infty.$$

Let f be defined on a left-hand open interval of $a \in \mathbb{R}$ (i.e. on $(a-\eta, a)$ for some $\eta > 0$). Write out the K - δ definition for

$$\lim_{x \to a-} f(x) = -\infty.$$

ii) Let f be defined for all sufficiently large positive x. Write out the K - X definitions for each of the following limits,

$$\lim_{x \to +\infty} f(x) = +\infty, \qquad \lim_{x \to +\infty} f(x) = -\infty,$$

iii) Let f be defined for all sufficiently large negative x. Write out the K - X definitions for each of the following limits.

$$\lim_{x \to -\infty} f(x) = +\infty, \qquad \lim_{x \to -\infty} f(x) = -\infty.$$

Solution i. The K - δ definitions of one-sided limits being infinite are

$$\lim_{x \to a+} f(x) = +\infty : \forall K > 0, \exists\, \delta > 0, \forall x : a < x < a + \delta \implies f(x) > K.$$

$$\lim_{x \to a-} f(x) = -\infty : \forall K < 0, \exists\, \delta > 0, \forall x : a - \delta < x < a \implies f(x) < K.$$

ii. The K - X definitions of limits at $+\infty$ being infinite are

$$\lim_{x \to +\infty} f(x) = +\infty : \forall K > 0, \exists\, X > 0, \forall x : x > X \implies f(x) > K.$$

$$\lim_{x \to +\infty} f(x) = -\infty : \forall K < 0, \exists\, X > 0, \forall x : x > X \implies f(x) < K.$$

iii. The K - X definitions of limits at $-\infty$ being infinite are

$$\lim_{x \to -\infty} f(x) = +\infty : \forall K > 0, \exists X < 0, \forall x : x < X \implies f(x) > K.$$

$$\lim_{x \to -\infty} f(x) = -\infty : \forall K < 0, \exists X < 0, \forall x : x < X \implies f(x) < K.$$

2. i) Write

$$G(x) = \frac{x}{x^2 - 1}$$

as partial fractions for $x \neq 1$ or -1.

ii) Prove that if $x > 1$ then

$$G(x) > \frac{1}{2\,(x-1)}.$$

Thus verify the K - δ definition (seen in Question 1i) of

$$\lim_{x \to 1+} G(x) = +\infty.$$

iii) Prove, that if $0 < x < 1$ then

$$G(x) \leq \frac{1}{2\,(x-1)} + \frac{1}{2}.$$

Thus show that the K - δ definition (seen in Question 1i) of

$$\lim_{x \to 1-} G(x) = -\infty$$

is verified by choosing $\delta = \min\left(1, -1/(2K-1)\right)$ for any given $K < 0$.

iv) Evaluate (so there is no need to verify the definition)

$$\lim_{x \to -1+} G(x) \quad \text{and} \quad \lim_{x \to -1-} G(x)\,.$$

v) Evaluate

$$\lim_{x \to +\infty} G(x) \quad \text{and} \quad \lim_{x \to -\infty} G(x)\,,$$

if they exist.

vi) Sketch the graph of G.

Solution i) The Partial Fraction is found starting from

$$\frac{x}{x^2-1} = \frac{A}{x-1} + \frac{B}{x+1}.$$

You may not have previously seen the following method to find the unknown A and B. Multiply up by $x-1$ to get

$$A + \frac{B(x-1)}{x+1} = \frac{x(x-1)}{(x^2-1)} = \frac{x}{x+1}.$$

Let $x \to 1$ to get $A = 1/2$. Similarly you can get $B = 1/2$. Thus

$$G(x) = \frac{1}{2}\left\{\frac{1}{x-1} + \frac{1}{x+1}\right\}.$$

Note in the next two parts we look at the values of $G(x)$ as $x \to 1$ from above and below. Of the two terms in the partial fraction form of $G(x)$ it is the $1/(x-1)$ term that dominates for such x. We are left to simply bound the remaining factor $1/(x+1)$.

ii) To show the limit is $+\infty$ we have to show that $G(x)$ is larger than any given $K > 0$, which we do by looking for *lower bounds* for $G(x)$.

If $x > 1$ then $x+1$ is positive in which case

$$\frac{1}{1+x} \quad \text{is positive, i.e.} \quad \frac{1}{1+x} > 0.$$

Thus we have a lower bound for G:

$$G(x) = \frac{1}{2}\left\{\frac{1}{x-1} + \frac{1}{x+1}\right\} > \frac{1}{2}\left\{\frac{1}{x-1} + 0\right\} = \frac{1}{2(x-1)}. \tag{1}$$

Let $K > 0$ be given, choose $\delta = 1/(2K) > 0$ and assume $1 < x < 1+\delta$. Then $0 < x-1 < \delta$ in which case

$$\frac{1}{x-1} > \frac{1}{\delta},$$

and hence, from (1) ,

$$G(x) > \frac{1}{2(x-1)} > \frac{1}{2\delta} = \frac{1}{2(1/(2K))} = K.$$

Thus we have verified the K - δ definition (seen in Question 1) of the one-sided limit

$$\lim_{x\to 1+} G(x) = +\infty.$$

iii) To show the limit is $-\infty$ we have to show that $G(x)$ is less than any given $K < 0$, which we do by looking for *upper bounds* for $G(x)$. Given $K < 0$ we are told to take $\delta = \min(1, 1/(1-2K))$. Assume $1 - \delta < x < 1$. then, since $\delta \leq 1$, we have $0 < x < 1$ and thus $1 < x + 1 < 2$ and

$$\frac{1}{2} < \frac{1}{x+1} < 1.$$

Thus we have an upper bound for G:

$$G(x) = \frac{1}{2}\left\{\frac{1}{x-1} + \frac{1}{x+1}\right\} < \frac{1}{2}\left\{\frac{1}{x-1} + 1\right\} = \frac{1}{2(x-1)} + \frac{1}{2}. \quad (2)$$

Next $\delta < 1/(1-2K)$ implies that

$$1 > x > 1 - \delta > 1 - \frac{1}{1-2K} = -\frac{2K}{1-2K}.$$

Hence

$$0 > x - 1 > -\frac{2K}{1-2K} - 1 = -\frac{1}{1-2K},$$

which, inverted, gives

$$\frac{1}{x-1} < -(1-2K).$$

Substituting back into (2) we find, for $1 - \delta < x < 1$,

$$G(x) \leq \frac{1}{2}\{-(1-2K) + 1\} = K,$$

as required. Hence we have verified the K - δ definition of

$$\lim_{x \to 1-} G(x) = -\infty.$$

iv) Without detailed proofs note that for x close to -1 it is the term $1/(x+1)$ in the partial expansion of $G(x)$ that is unbounded. The other term, $1/(x-1)$, will be bounded.

For the **right hand limit** at -1, if $-1 < x < 0$ then $x + 1 > 0$, i.e. is positive. So $1/(x+1)$ will become arbitrarily large and *positive* as x approaches -1 from above and thus

$$\lim_{x \to -1+} G(x) = +\infty.$$

For the **left hand limit** at -1, if $x < -1$ then $x + 1 < 0$ i.e. is negative. Thus $1/(x+1)$ will become arbitrarily large and *negative* as x approaches -1 from below and hence

$$\lim_{x \to -1-} G(x) = -\infty.$$

EXTRA Though you were not asked in the question to verify the K - δ definitions of the last two limits we do so here.

For $\mathbf{x \to -1+}$ let $K > 0$ be given, choose $\delta = \min(1, 1/(2K+1)) > 0$ and assume $-1 < x < -1 + \delta$. Since $\delta \le 1$ we have $-1 < x < 0$, i.e. $-2 < x - 1 < -1$ in which case

$$-\frac{1}{2} > \frac{1}{x-1} > -1.$$

But $-1 < x < -1 + \delta$ also implies $0 < x + 1 < \delta$, in which case

$$\frac{1}{x+1} > \frac{1}{\delta}.$$

Combine these lower bounds in

$$\begin{aligned} G(x) &= \frac{1}{2}\left\{\frac{1}{x-1} + \frac{1}{x+1}\right\} > \frac{1}{2}\left\{-1 + \frac{1}{\delta}\right\} \\ &\ge \frac{1}{2}\left\{-1 + \frac{1}{1/(2K+1)}\right\} \quad \text{since } \delta \le 1/(2K+1) \\ &= K. \end{aligned}$$

Thus, for all $K > 0$ we can find a $\delta > 0$ such that if $-1 < x < -1 + \delta$ then $G(x) \geq K$. This is the K - δ definition of $\lim_{x \to -1+} G(x) = +\infty$.

For $\mathbf{x \to -1-}$ let $K < 0$ be given, choose $\delta = -1/(2K) > 0$ and assume $-1 - \delta < x < -1$. Then, *with no restriction from* δ we have $x - 1 < -2$ in which case

$$-\frac{1}{2} < \frac{1}{x-1} < 0.$$

But $-1 - \delta < x < -1$ also implies $-\delta < x + 1 < 0$ in which case

$$\frac{1}{x+1} < -\frac{1}{\delta}.$$

Combine these upper bounds in

$$G(x) = \frac{1}{2}\left\{\frac{1}{x-1} + \frac{1}{x+1}\right\} \leq \frac{1}{2}\left\{0 + \left(-\frac{1}{\delta}\right)\right\} = K.$$

Thus, for all $K < 0$ we can find a $\delta > 0$ such that if $-1 - \delta < x < -1$ then $G(x) \leq K$. This is the K - δ definition of $\lim_{x \to -1-} G(x) = -\infty$.

(v) For large x the function $G(x)$ "looks like"

$$\frac{x}{x^2} = \frac{1}{x}.$$

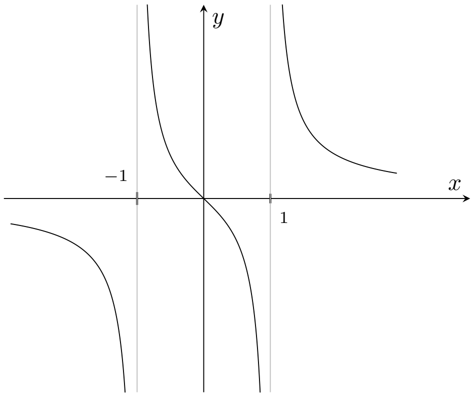
Hence, without detailed proofs, we can still say that the limits exist and

$$\lim_{x \to +\infty} G(x) = \lim_{x \to -\infty} G(x) = 0.$$

vi) The graph of G is

3. Follow the example in the notes, $\lim_{x \to 1} x/(x-1)^2 = \infty$, to verify the K - δ definitions of

$$\text{i)} \lim_{x \to -3} \frac{x^2}{(x+3)^2} = +\infty \quad \text{and} \quad \text{ii)} \lim_{x \to -3} \frac{x}{(x+3)^2} = -\infty.$$



Solution i) To show the limit is $+\infty$ we have to show the function is larger than any given $K > 0$, which we do by looking for *lower bounds* for the function.

Let $K > 0$ been given, choose $\delta = \min\left(1, 2/\sqrt{K}\right)$. and assume $0 < |x+3| < \delta$.

Then

$$\begin{aligned} \delta \le 1 \quad \text{and} \quad 0 < |x+3| < \delta \quad &\Longrightarrow \quad -4 < x < -2 \\ &\Longrightarrow \quad 4 < x^2 < 16 \qquad (3) \\ &\Longrightarrow \quad \frac{x^2}{(x+3)^2} > \frac{4}{(x+3)^2}. \end{aligned}$$

having divided the first inequality of (3) by the positive $(x+3)^2$. Next

$$\begin{aligned} \delta \le \frac{2}{\sqrt{K}} \quad \text{and} \quad 0 < |x+3| < \delta \quad &\Longrightarrow \quad (x+3)^2 \le \frac{4}{K} \\ &\Longrightarrow \quad \frac{4}{(x+3)^2} \ge K. \end{aligned}$$

Hence $\delta = \min\left(1, 2/\sqrt{K}\right)$ and $0 < |x+3| < \delta$ together imply

$$\frac{x^2}{(x+3)^2} > \frac{4}{(x+3)^2} \ge K$$

Thus we have verified the K - δ definition of

$$\lim_{x \to -3} \frac{x^2}{(x+3)^2} = +\infty.$$

ii) To show the limit is $-\infty$ we have to show the function is less than any given $K < 0$, which we do by looking for *upper bounds* for the function.

Let $K < 0$ be given. Choose $\delta = \min\left(1, \sqrt{-2/K}\right) > 0$. Note that because $K < 0$ we have $-2/K > 0$ and we can take the square root. Assume $0 < |x+3| < \delta$.

First,

$$\delta \leq 1 \quad \text{and} \quad 0 < |x+3| < \delta \implies -4 < x < -2 \tag{4}$$

$$\implies \frac{x}{(x+3)^2} < -\frac{2}{(x+3)^2}, \tag{5}$$

having divided the first inequality of (4) by the positive $(x+3)^2$. Next

$$\delta \leq \sqrt{-\frac{2}{K}} \quad \text{and} \quad 0 < |x+3| < \delta \implies (x+3)^2 < -\frac{2}{K}$$

$$\implies \frac{1}{(x+3)^2} > -\frac{K}{2}$$

$$\implies -\frac{2}{(x+3)^2} < K. \tag{6}$$

Combining (5) and (6) we have, for $\delta = \min\left(1, \sqrt{-2/K}\right)$ and $0 < |x+3| < \delta$, that

$$\frac{x}{(x+3)^2} < -\frac{2}{(x+3)^2} < K.$$

Thus we have verified the K - δ definition of

$$\lim_{x \to -3} \frac{x}{(x+3)^2} = -\infty.$$

4. Define $H : \mathbb{R} \to \mathbb{R}$ by

$$H(x) = \frac{1}{x^2+1} + x.$$

Prove by verifying the K - X definitions that

$$\lim_{x \to +\infty} H(x) = +\infty \quad \text{and} \quad \lim_{x \to -\infty} H(x) = -\infty.$$

Sketch the graph of H.

Solution To prove $\lim_{x\to+\infty} H(x) = +\infty$, let $K > 0$ be given. Choose $X = K$.

Assume $x > X$.

Remember, we hope to prove $H(x) > K$ so we look for *lower* bounds on $H(x)$. For the present result it suffices to note that

$$H(x) = \frac{1}{x^2+1} + x > x,$$

where we are simplifying the expression by "throwing away" the complicated part $1/(x^2+1) > 0$. Continuing,

$$H(x) > x > X = K.$$

Thus we have verified the K - X definition of $\lim_{x\to+\infty} H(x) = +\infty$.

To prove $\lim_{x\to-\infty} H(x) = -\infty$ let $K < 0$ be given. Choose $X = K - 1$.

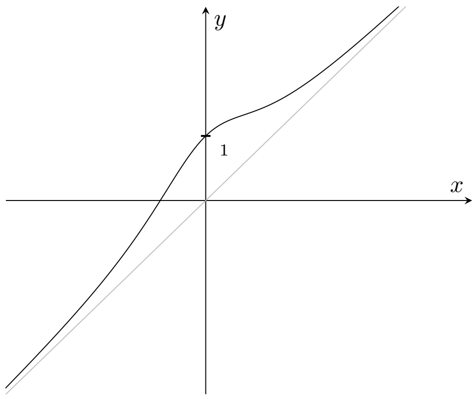
Assume $x < X$.

We hope to prove $H(x) < K$ and so we look for *upper* bounds on $H(x)$. This means that we cannot simply throw away the $1/(x^2+1)$ term. Instead we use the fact that $1/(x^2+1) < 1$ for any $x \in \mathbb{R}$. Then

$$H(x) = \frac{1}{x^2+1} + x < 1 + x < 1 + X = K,$$

by the choice of X. Thus we have verified the K - X definition of $\lim_{x\to-\infty} H(x) = -\infty$.

The graph of $H(x)$ is



Limit Rules

5. Using the **Limit Rules** evaluate

i)

$$\lim_{x\to 0} \frac{3x^2+4x+1}{x^2+4x+3},$$

ii)

$$\lim_{x\to \infty} \frac{3x^2+4x+1}{x^2+4x+3},$$

iii)

$$\lim_{x\to -1} \frac{3x^2+4x+1}{x^2+4x+3}.$$

Note When using a Limit Rule you **must** write down which Rule you are using and you **must** show that any necessary conditions of that rule are satisfied.

Solution i) The rational function

$$\frac{3x^2+4x+1}{x^2+4x+3}$$

is well-defined at 0 (in particular the denominator is not 0) so by the Quotient Rule for limits

$$\lim_{x\to 0} \frac{3x^2+4x+1}{x^2+4x+3} = \frac{\lim_{x\to 0}(3x^2+4x+1)}{\lim_{x\to 0}(x^2+4x+3)} = \frac{1}{3}.$$

ii) We cannot apply the Quotient Rule for limits directly since the polynomials on the numerator and denominator diverge as $x \to +\infty$. Instead, divide top and bottom by the largest power of x to get

$$\begin{aligned} \lim_{x\to+\infty} \frac{3x^2+4x+1}{x^2+4x+3} &= \lim_{x\to+\infty} \frac{3+4/x+1/x^2}{1+4/x+3/x^2} \\ &= \frac{\lim_{x\to+\infty}(3+4/x+1/x^2)}{\lim_{x\to+\infty}(1+4/x+3/x^2)} \qquad (7) \\ &= \frac{3}{1} = 3. \end{aligned}$$

Here we have used the Quotient Rule at (7), allowable since both limits exist and the one on the denominator is non-zero.

iii) We cannot apply the Quotient Rule for limits since the denominator is 0 at $x = -1$. This means that the denominator has a factor of $x + 1$ and in fact

$$x^2 + 4x + 3 = (x + 1)(x + 3).$$

For the limit of the rational function to exist the numerator will also have to be zero at $x = -1$, i.e. have a factor of $x + 1$. In fact

$$3x^2 + 4x + 1 = (x + 1)(3x + 1).$$

Thus

$$\begin{aligned} \lim_{x\to-1} \frac{3x^2+4x+1}{x^2+4x+3} &= \lim_{x\to-1} \frac{(x+1)(3x+1)}{(x+1)(x+3)} \\ &= \lim_{x\to-1} \frac{3x+1}{x+3}. \end{aligned}$$

We can now apply the Quotient Rule for limits since both $\lim_{x\to-1}(3x + 1)$ and $\lim_{x\to-1}(x + 3)$ exist and the second one is non-zero. Hence

$$\lim_{x\to-1} \frac{3x^2+4x+1}{x^2+4x+3} = \frac{\lim_{x\to-1}(3x+1)}{\lim_{x\to-1}(x+3)} = \frac{-2}{2} = -1.$$

6. (i) What is wrong with the argument:

$$\begin{aligned}\lim_{x\to 0} x^3 \sin\left(\frac{\pi}{x}\right) &= \lim_{x\to 0} x^3 \times \lim_{x\to 0} \sin\left(\frac{\pi}{x}\right) \\ &\qquad \text{by the Product Rule} \\ &= 0 \times \lim_{x\to 0} \sin\left(\frac{\pi}{x}\right) \\ &= 0.\end{aligned}$$

(ii) Evaluate

$$\lim_{x\to 0} x^3 \sin\left(\frac{\pi}{x}\right).$$

Solution i) You may **only** use the Product Rule for limits when both individual limits exist. Here we know from Question 1 Sheet 2 that $\lim_{x\to 0} \sin(\pi/x)$ does **not** exist, so we cannot apply the Product Rule (even if the answer it gives is correct!)

ii) We might guess that the limit is 0.

Let $\varepsilon > 0$ be given, choose $\delta = \varepsilon^{1/3}$ and assume $x : 0 < |x - 0| < \delta$. Then

$$\begin{aligned}\left|x^3 \sin\left(\frac{\pi}{x}\right) - 0\right| = \left|x^3 \sin\left(\frac{\pi}{x}\right)\right| &\leq \left|x^3\right| && \text{since } |\sin(\pi/x)| \leq 1, \\ &= |x|^3 < \delta^3 && \text{since } |x - 0| < \delta \\ &< \left(\varepsilon^{1/3}\right)^3 = \varepsilon && \text{since } \delta = \varepsilon^{1/3}.\end{aligned}$$

Hence we have verified the definition of

$$\lim_{x\to 0} x^3 \sin\left(\frac{\pi}{x}\right) = 0.$$

Alternatively you could use the Sandwich Rule on

$$-|x|^3 \leq x^3 \sin\left(\frac{\pi}{x}\right) \leq |x|^3.$$

Exponential and trigonometric examples

7. Recall that in the lectures we have shown that

$$\lim_{x\to 0} e^x = 1 \quad \text{and} \quad \lim_{x\to 0} \frac{e^x - 1}{x} = 1.$$

Use these to evaluate the following limits which include the hyperbolic functions.

(i)

$$\lim_{x\to 0} \frac{\sinh x}{x},$$

ii)

$$\lim_{x\to 0} \frac{\tanh x}{x},$$

iii)

$$\lim_{x\to 0} \frac{\cosh x - 1}{x^2}.$$

Solution i) Start from

$$\frac{\sinh x}{x} = \frac{e^x - e^{-x}}{2x}.$$

The guiding principle is to manipulate this so we see a function whose limit we already know. For example $(e^x - 1)/x$. For this reason we 'add in zero' in the form $0 = -1 + 1$:

$$\begin{aligned} \frac{\sinh x}{x} &= \frac{e^x - 1 + 1 - e^{-x}}{2x} = \frac{1}{2}\left(\frac{e^x - 1}{x}\right) + \frac{e^{-x}}{2}\left(\frac{e^x - 1}{x}\right) \\ &= \frac{1}{2}\left(\frac{e^x - 1}{x}\right) + \frac{1}{2e^x}\left(\frac{e^x - 1}{x}\right). \end{aligned}$$

Now use the Sum and Product Rules for limits to get

$$\begin{aligned} \lim_{x\to 0} \frac{\sinh x}{x} &= \frac{1}{2}\lim_{x\to 0}\left(\frac{e^x - 1}{x}\right) + \frac{1}{2\lim_{x\to 0} e^x}\lim_{x\to 0}\left(\frac{e^x - 1}{x}\right) \\ &= \frac{1}{2} + \frac{1}{2} = 1. \end{aligned}$$

ii) With the intention of using known results write

$$\frac{\tanh x}{x} = \frac{\sinh x}{x} \times \frac{1}{\cosh x}.$$

Before we apply the Quotient Rule for limits we need to note that

$$\cosh x = \frac{e^x + e^{-x}}{2} = \frac{1}{2}\left(e^x + \frac{1}{e^x}\right) \longrightarrow \frac{1}{2}\left(1 + \frac{1}{1}\right) = 1,$$

as $x \to 0$. Because this exists and is non-zero we can apply the Quotient Rule to get

$$\lim_{x\to 0} \frac{\tanh x}{x} = \frac{\lim_{x\to 0} \frac{\sinh x}{x}}{\lim_{x\to 0} \cosh x} = \frac{1}{1} = 1.$$

We have used Part i in the numerator.

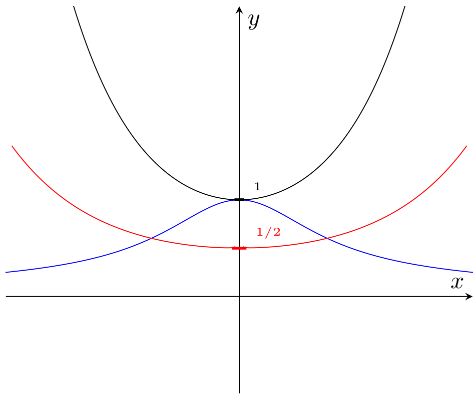
iii) Apply the same idea of 'multiplying by 1' as used for $(\cos x - 1)/x^2$ in lectures: For $x \neq 0$,

$$\begin{aligned}\frac{\cosh x - 1}{x^2} &= \frac{\cosh x - 1}{x^2} \times \left(\frac{\cosh x + 1}{\cosh x + 1}\right) = \frac{\cosh^2 x - 1}{x^2 (\cosh x + 1)} \\ &= \left(\frac{\sinh x}{x}\right)^2 \frac{1}{\cosh x + 1} \qquad \text{since } \cosh^2 x - \sinh^2 x = 1, \\ &\longrightarrow 1^2 \times \frac{1}{2} \quad \text{as } x \to 0,\end{aligned}$$

by the Product and Quotient Rules and Part i. Thus

$$\lim_{x\to 0} \frac{\cosh x - 1}{x^2} = \frac{1}{2}.$$

The graphs of these functions are not particularly interesting, but I have plotted the graph of $y = \sinh x/x$ in black, $y = \tanh x/x$ in blue and of $y = (\cosh x - 1)/x^2$ in red:



8. i) Assuming that $e^x > x$ for all $x > 0$ verify the ε - X definitions of

$$\lim_{x\to+\infty} e^{-x} = 0 \quad \text{and} \quad \lim_{x\to-\infty} e^{x} = 0.$$

Deduce (using the Limit Rules) that

$$\lim_{x\to+\infty} \tanh x = 1 \quad \text{and} \quad \lim_{x\to-\infty} \tanh x = -1.$$

Sketch the graph of $\tanh x$.

Solution i) Let $\varepsilon > 0$ be given. Choose $X = 1/\varepsilon > 0$. Assume $x > X$. By the assumption in the question we have $e^x > x$ so

$$0 < e^{-x} = \frac{1}{e^x} < \frac{1}{x} < \frac{1}{X} = \frac{1}{(1/\varepsilon)} = \varepsilon.$$

Thus we have verified the ε - X definition of $\lim_{x\to+\infty} e^{-x} = 0$.

Let $\varepsilon > 0$ be given. Choose $X = -1/\varepsilon < 0$. Assume $x < X$. This means that x is negative, so can be written as $x = -y$ where $y > -X = 1/\varepsilon$. Then, as above,

$$e^x = e^{-y} < \frac{1}{y} < \frac{1}{(-X)} = \frac{1}{(1/\varepsilon)} = \varepsilon.$$

Thus we have verified the ε - X definition of $\lim_{x\to-\infty} e^{x} = 0$.

ii) By definition

$$\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}}.$$

- For $x \to +\infty$ divide top and bottom by e^x so

$$\tanh x = \frac{1 - e^{-2x}}{1 + e^{-2x}}.$$

By the Product Rule for limits, part i of this question gives

$$\lim_{x \to +\infty} e^{-2x} = \lim_{x \to +\infty} \left(e^{-x}\right)^2 = \left(\lim_{x \to +\infty} e^{-x}\right)^2 = 0.$$

Then, by the Quotient Rule for limits,

$$\lim_{x \to +\infty} \tanh = \frac{\lim_{x \to +\infty} \left(1 - e^{-2x}\right)}{\lim_{x \to +\infty} \left(1 + e^{-2x}\right)} = 1.$$

- For $x \to -\infty$ divide top and bottom by e^{-x} so

$$\tanh x = \frac{e^{2x} - 1}{e^{2x} + 1}.$$

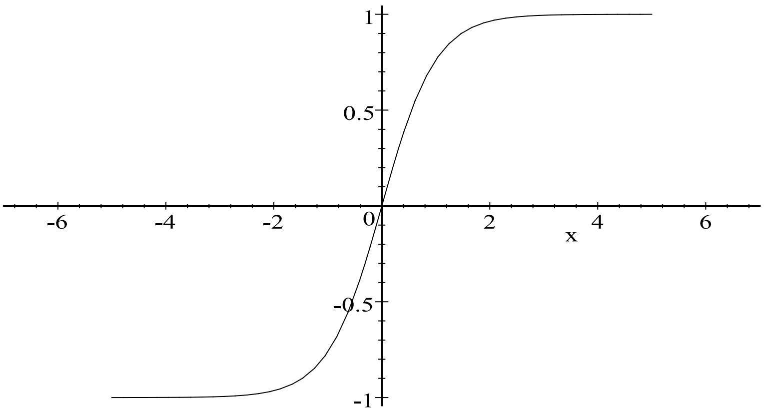
Again the Product Rule for limits and part i gives

$$\lim_{x \to -\infty} e^{2x} = 0.$$

Then, by the Quotient Rule for limits,

$$\lim_{x \to +\infty} \tanh = \frac{\lim_{x \to +\infty} \left(e^{2x} - 1\right)}{\lim_{x \to +\infty} \left(e^{2x} + 1\right)} = 1.$$

Finally, we can use the results just found to plot the graph of $y = \tanh x$:



Additional Questions

9. i. Prove that

$$\left| e^x - 1 - x - \frac{x^2}{2} - \frac{x^3}{6} \right| < \frac{2}{4!} \left| x^4 \right|$$

for $|x| < 1/2$.

Hint Use the method seen in the notes where it was shown that $|e^x - 1 - x| < |x^2|$ for $|x| < 1/2$.

ii. Deduce

$$\lim_{x \to 0} \frac{e^x - 1 - x - x^2/2}{x^3} = \frac{1}{6}.$$

iii. Use Part ii. to evaluate

$$\lim_{x \to 0} \frac{\sinh x - x}{x^3}.$$

Solution i) Start from the definition of an infinite series as the limit of the sequence of partial sums, so

$$e^x - 1 - x - \frac{x^2}{2} - \frac{x^3}{3!} = \lim_{N \to \infty} \sum_{k=4}^{N} \frac{x^k}{k!} = x^4 \lim_{N \to \infty} \sum_{j=0}^{N-4} \frac{x^j}{(j+4)!}. \qquad (8)$$

Then, by the triangle inequality, (applicable since we have a **finite** sum),

$$\left|\sum_{j=0}^{N-4}\frac{x^j}{(j+4)!}\right| \le \sum_{j=0}^{N-4}\frac{|x|^j}{(j+4)!} \le \frac{1}{4!}\sum_{j=0}^{N-4}|x|^j$$

since $(j+4)! \ge 4!$ for all $j \ge 0$,

$$= \frac{1}{4!}\left(\frac{1-|x|^{N-3}}{1-|x|}\right),$$

on summing the Geometric Series, allowable when $|x| \neq 1$. In fact we have $|x| < 1/2 < 1$, which means

$$\frac{1-|x|^{N-3}}{1-|x|} \le \frac{1}{1-|x|} < \frac{1}{1-1/2} = 2.$$

Hence

$$\left|\sum_{j=0}^{N-3}\frac{x^j}{(j+4)!}\right| \le \frac{2}{4!}$$

for all $N \ge 0$. Therefore, *since* the limit of these partial sums exists the limit must satisfy

$$\left|\lim_{N\to\infty}\sum_{j=0}^{N-3}\frac{x^j}{(j+4)!}\right| \le \frac{2}{4!}.$$

Combined with (8) we have

$$\left|e^x - 1 - x - \frac{x^2}{2} - \frac{x^3}{3!}\right| \le \frac{2}{4!}|x|^4.$$

ii) Divide through the result of part i by $|x^3|$ to get

$$\left|\frac{e^x - 1 - x - x^2/2}{x^3} - \frac{1}{6}\right| < \frac{2}{4!}|x| < |x| \tag{9}$$

for $|x| < 1/2$.

To prove the limit in the question we can verify the definition. Let $\varepsilon > 0$ be given, choose $\delta = \min(1/2, \varepsilon)$ and assume $0 < |x - 0| < \delta$.

Since $\delta \leq 1/2$, the inequality (9) holds for such x. Thus

$$\left| \frac{e^x - 1 - x - x^2/2}{x^3} - \frac{1}{6} \right| < |x| < \delta \leq \varepsilon.$$

Hence we have verified the ε - δ definition of

$$\lim_{x \to 0} \frac{e^x - 1 - x - x^2/2}{x^3} = \frac{1}{6}. \tag{10}$$

Alternatively we can use the Sandwich Rule for (9) opens out as

$$\frac{1}{6} - |x| < \frac{e^x - 1 - x - x^2/2}{x^3} < \frac{1}{6} + |x| .$$

Let $x \to 0$ when the upper and lower bound $\to 1/6$. Thus, by the Sandwich Rule, (10) follows.

iii) From the definition of $\sinh x$ we have

$$\frac{\sinh x - x}{x^3} = \frac{e^x - e^{-x} - 2x}{2x^3}.$$

This has to be manipulated so that we see $e^x - 1 - x - x^2/2$ and can thus use (10). Do this by "adding in zero" in the form

$$0 = -x^2/2 - \left(- (-x)^2 2 \right),$$

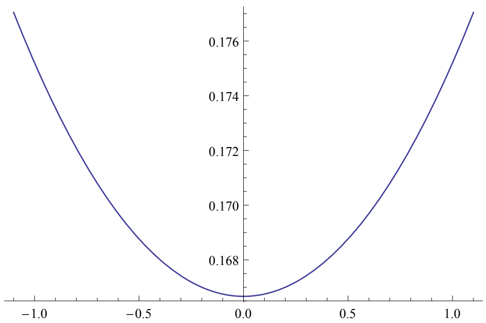
to get

$$\begin{aligned} \frac{e^x - e^{-x} - 2x}{2x^3} &= \frac{(e^x - 1 - x - x^2/2) - (e^{-x} - 1 - (-x) - (-x)^2/2)}{2x^3} \\ &= \frac{(e^x - 1 - x - x^2/2)}{2x^3} + \frac{(e^{-x} - 1 - (-x) - (-x)^2/2)}{2(-x)^3}. \end{aligned}$$

Let $x \to 0$ (in which case $-x \to 0$) when, by the assumption of the question, we get

$$\begin{aligned}\lim_{x\to 0}\frac{\sinh x - x}{x^3} &= \frac{1}{2}\lim_{x\to 0}\frac{\left(e^x - 1 - x - x^2/2\right)}{x^3} \\ &\qquad + \frac{1}{2}\lim_{-x\to 0}\frac{\left(e^{-x} - 1 - (-x) - (-x)^2/2\right)}{2(-x)^3} \\ &= \frac{1}{2}\times\frac{1}{6} + \frac{1}{2}\times\frac{1}{6} = \frac{1}{6}.\end{aligned}$$

Again, the graph of $y = (\sinh x - x)/x^3$ is not particularly 'exciting':



10. Recall that in the lectures we have shown that

$$\lim_{\theta\to 0}\frac{\sin\theta}{\theta} = 1$$

Use this to evaluate (**without** using L'Hôpital's Rule)

i)

$$\lim_{\theta\to 0}\frac{\theta}{\tan\theta},$$

ii)

$$\lim_{\theta\to 0}\frac{\sin\theta - \tan\theta}{\theta^3}.$$

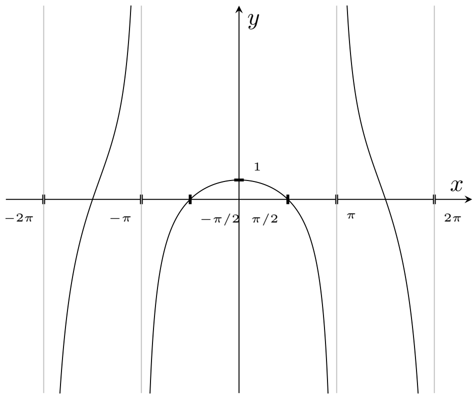
Solution i) Again guided by the limits we already know write

$$\lim_{\theta\to 0}\frac{\theta}{\tan\theta} = \lim_{\theta\to 0}\frac{\theta\cos\theta}{\sin\theta} = \lim_{\theta\to 0}\frac{\cos\theta}{\left(\frac{\sin\theta}{\theta}\right)} = \frac{\lim_{\theta\to 0}\cos\theta}{\lim_{\theta\to 0}\frac{\sin\theta}{\theta}},$$

by Quotient Rule for limits, allowable since both limits exist and the limit on the denominator is non-zero. Hence

$$\lim_{\theta\to 0}\frac{\theta}{\tan\theta}=\frac{1}{1}=1.$$

Graphically, $y = x/\tan x$:



ii) The limit we already know from lectures is of $(\cos\theta - 1)/\theta^2$ so write

$$\frac{\sin\theta-\tan\theta}{\theta^3}=\frac{\tan\theta}{\theta}\left(\frac{\cos\theta-1}{\theta^2}\right).$$

The "trick" used in lectures to evaluate the limit of this it is to multiply top and bottom by $\cos\theta + 1$ to get

$$\begin{aligned}\frac{\tan\theta}{\theta}\left(\frac{\cos^2\theta-1}{\theta^2}\right)\frac{1}{\cos\theta+1} &= -\frac{\tan\theta}{\theta}\left(\frac{\sin\theta}{\theta}\right)^2\frac{1}{\cos\theta+1}\\ &= -\frac{1}{\cos\theta}\left(\frac{\sin\theta}{\theta}\right)^3\frac{1}{\cos\theta+1}.\end{aligned}$$

Use the Product and Quotient Rules for limits to deduce

$$\lim_{\theta\to 0}\frac{\sin\theta-\tan\theta}{\theta^3}=-\frac{1}{2}.$$

Graphically, $y = (\sin x - \tan x) / x^2$: